# A model of sedimentary delta growth: a novel application of numerical heat transfer methods

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**Abstract**

**Purpose** – Understanding the factors that contribute to the growth of sediment delta lobes in river systems has significant benefit towards protecting civil and social infrastructure from severe weather events. To develop this understanding, this paper aims to construct a three-dimensional numerical model of a sediment delta depositing on to a two-dimensional bedrock basement entering an ocean at a constant sea-level.

**Design/methodology/approach** – The approach used adapts and applies techniques and schemes previously used in building numerical heat transfer models of melting systems. Particular emphasis is placed on modifying fixed grid enthalpy like schemes.

**Findings** – The resulting model provides important insight on the features that control the partition of sediment delta deposition between the land and ocean domains. The model also illustrates how tectonic subsidence may control the rate of delta growth.

**Originality/value** – This is the first numerical heat transfer inspired model of a three-dimensional sediment delta deposit over both land and ocean domains. The problem has scientific merit in that it represents a melting-like moving boundary problem with two distinct moving boundaries and a space/time dependent latent heat. Further, this work is a necessary first step towards building a comprehensive understanding of how to restore delta systems to protect civil and social infrastructure.

**Keywords** Numerical analysis, Heat transfer, Sedimentation, Coastal regions, Landforms, Modelling

**Paper type** Research paper

## Nomenclature

| | |
|---|---|
| *Flux* | flux across the face of a control volume |
| $H$ | enthalpy |
| $h$ | height of sediment above sea-level |
| $L$ | latent heat (ocean depth) |
| $\ell$ | length scale |
| $\mathbf{n}$ | outward pointing normal |
| $\mathbf{q}$ | sediment unit flux |
| $q_0$ | component of input unit sediment flux in $x$-direction |
| $s$ | one-dimensional position of shoreline |
| $t$ | time |
| $\mathbf{v}$ | front velocity |
| $w$ | width of sediment input |

*Greek letters*

| | |
|---|---|
| $\beta$ | bedrock slope |
| $\dot{\beta}$ | rate of increase in bedrock slope |
| $\delta t$ | time step |
| $\varepsilon$ | deposit porosity |
| $\eta$ | elevation of bedrock |

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$\nu$ diffusivity

*Superscript*

*new* new time level

*Subscripts*

*N, S, E, W* face positions

*i, j* row and column index



## Introduction

The 2005 Gulf of Mexico hurricanes Katrina and Rita provided a clear illustration of the negative impact of landscape degradation on the sustainability of costal social and civil infrastructures. Due to a combination of a decrease in river sediment supply, channel confinement and disconnection from the floodplain, sea-level rise, and land sinking (subsidence), the Mississippi Deltaic Plain is disappearing at rates estimated on the order of 44 square kilometres (17 square miles) per year. In addition to supplying a diverse, productive habitat these off-cost lands also provide an important buffer from severe storms for the city of New Orleans and other coastal population centres. An ambitious but feasible plan, to reverse land loss and provide protection, is to breach the Mississippi river levees downstream of New Orleans thereby diverting a significant sediment load for delta land building (Kim *et al.*, 2009). The success of this project will hinge on developing a comprehensive understanding how delta lobes and their associated ecosystems grow (Day *et al.*, 2007). Towards this end, as summarized in the literature review below, a number of recent research works have been directed at building analytical and numerical models that can be used to describe the deposit and growth of sediment lobes in river deltas. A key feature in these efforts has been to adopt and modify analysis approaches developed for heat transfer problems. A particularly fruitful line of research has been the application of numerical solutions of heat conduction controlled phase change problems for modelling the mass balance in a growing sediment delta. The majority of these efforts have focused on one-dimensional approaches that model the formation of a two-dimensional sediment wedge on the basement bedrock. To date, models in two-dimensional domains, that produce developing three-dimensional sediment cones, have been restricted to problems of point sources entering an ocean domain (Voller *et al.*, 2006). The object of this work is to extend the previous endeavourers and consider the formation of a three-dimensional depositing sediment cone that forms over both land and ocean domains.

## Description of an example problem

A representative example problem, that will drive the numerical developments in this work, is the building of a three-dimensional sediment deposit on a bedrock basement slope that enters into an ocean with a constant sea-level; Figure 1 shows the initial domain. Here, the bedrock slope $\beta$ is assumed constant and the origin is placed at the intersection of the sea-level and bedrock. In this way, the bedrock elevations can be calculated as $\eta = \beta x$ with the bedrock elevation at sea-level taking the value $\eta = 0$. The sediment deposit is formed by introducing (at time $t = 0$) a mixture of sediment and water along a strip of width $w$ ($-w/2 \le y \le w/2$) well inland of the ocean. When the resulting sediment line flux $\boldsymbol{q} = (q_0, 0)\ \mathrm{m^2 s^{-1}}$ reaches the ocean it is deposited and forms a cone. In plan view, the boundaries of this cone will advance in both the seaward and landward directions. A time snapshot of the three-dimensional above sea-level forming delta sediment deposit, measured as height $h$ above sea-level is shown in Figure 2(a). The sediment is deposited into the ocean to build an ocean delta with a shoreline boundary moving seaward. Initially, the sediment moves over the land

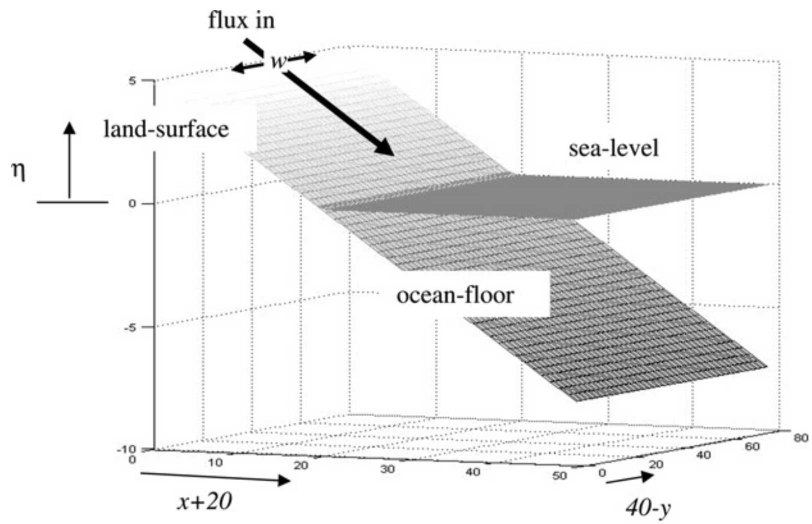



**Figure 1.**
Solution domain bedrock surface (land surface and ocean-floor) entering an ocean with a fixed sea-level

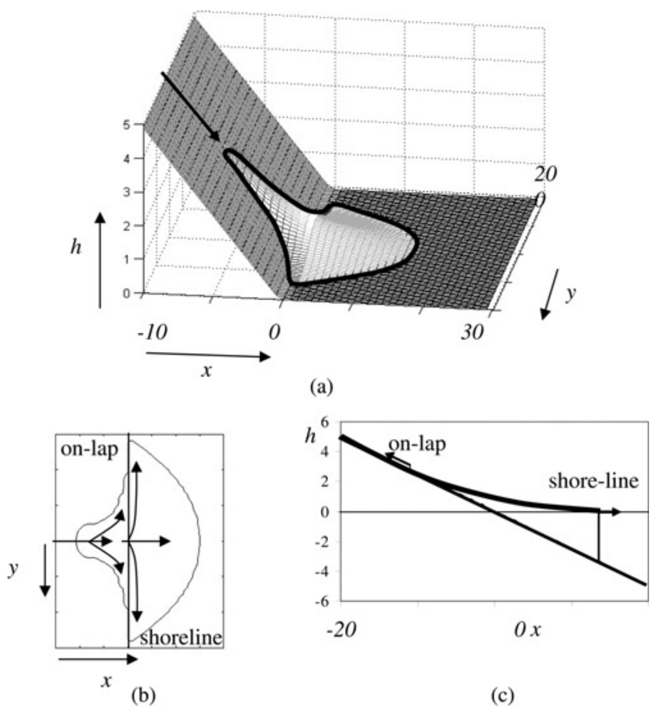



**Figure 2.**
Schematic of the delta building process

**Notes:** Time snap shots of (a) conical sediment deposit above sea-level; (b) plan form sediment deposit identifying shoreline and on-lap moving boundaries; and (c) mid section cross-section of sediment wedge surface with out depositing. As the sediment deposits in the ocean, however, the sediment cone also encroaches landward, forming an additional boundary – referred to as the on-lap – moving up the land slope. A time snap shot of the plan form of the sediment deposit, distinguishing the moving shoreline and on-lap boundaries, is shown in Figure 2(b). An additional perspective, also identifying the moving shoreline and on-lap boundaries, is given by the mid-plane cross-section of the deposit in Figure 2(c). With reference to this figure, it is noted that a typical slope for the submarine sediment deposit is significantly steeper that the slope of sub-aerial sediment, hence in calculations it is reasonable to assume that the submarine slope is vertical without loss of accuracy.



## The Swenson-Stefan analogy

The critical observation that allows for the use of numerical heat transfer tools to model the formation of sedimentary deltas was made by Swenson *et al.* (2000). These authors consider a one-dimensional model of a point sediment + water source issuing into a sloping ocean. A schematic of this geometry is shown in Figure 3. Here, for simplicity of treatment, it is assumed that:

- the submarine deposit has a vertical slope; and
- the bedrock basement slope remains fixed at $\beta$ (i.e. there are no tectonic actions).

In this way, the transport and deposit of the sediment in the system is governed by the Exner equation (see, Paola and Voller, 2005):

$$\frac{\partial h}{\partial t} = \frac{1}{\varepsilon}\frac{\partial q}{\partial x}, \qquad 0 \leq x \leq s(t), \tag{1}$$

where $s(t)$ is the moving shoreline position, $h(x, t)$ is the height of the sediment deposit above sea-level, $\varepsilon$ is the porosity of the sediment deposit, and $q$ is the sediment unit flux. Using a momentum balance coupled to basic sediment transport laws, it is possible to show that in field (Paola *et al.*, 1992) and laboratory settings (Lorenzo-Trueba *et al.*, 2009) it is reasonable to model the sediment flux with a diffusion law written in terms of the local slope of the sediment surface:

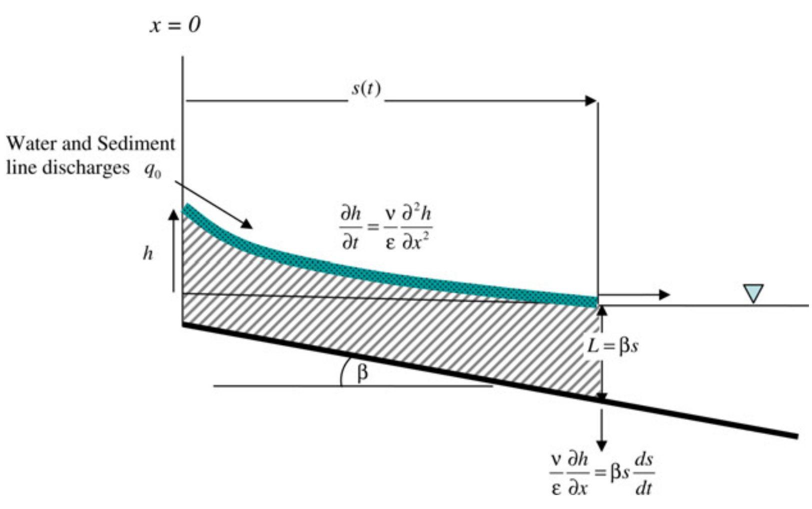



**Figure 3.** Schematic of one-dimensional sediment deposit into a sloping ocean that results in the Swenson-Stefan analogy

$$q = -\nu \frac{\partial h}{\partial x}, \tag{2}$$

where $\nu$, the diffusion coefficient ($m^2/s$), depends on such things as the water discharge and sediment properties (e.g. grain size). With the diffusion model, the Exner equation takes the familiar form of the Fourier heat conduction equation, i.e:

$$\frac{\partial h}{\partial t} = \frac{\nu}{\varepsilon} \frac{\partial^2 h}{\partial x^2}, \qquad 0 \leq x \leq s(t). \tag{3}$$

The problem is closed by specifying the initial conditions $s = 0,\ h = 0$ and defining the boundary conditions:

$$-\nu \frac{\partial h}{\partial x}\bigg|_{x=0} = q_o,$$

$$h(x = s, t) = 0, \tag{4}$$

$$-\frac{\nu}{\varepsilon}\frac{\partial h}{\partial x} = \beta s \frac{ds}{dt}.$$

The last boundary condition, needed to determine the movement of the shoreline $s(t)$, is essentially a balance between the surplus sediment arriving at the shoreline and the rate at which the ocean ahead of the shoreline can be filled. The critical observation made by Swenson *et al.* (2000) was to associate this mass balance condition to the classic the Stefan heat balance condition in a one-phase melting problem (Crank, 1984). This fact, coupled to the Fourier heat condition nature of the domain sediment transport Equation (3), allowed for the application of a substantial body of work on heat transfer melting for modelling the formation of sedimentary deltas. It is important to note, however, that the Swenson-Stefan condition in Equation (4) has an interesting element not seen in the classic Stefan condition. Namely, the latent heat term $L = \beta s$ is not a constant in space but increases with increasing shoreline position. This interesting feature can be compounded by allowing for tectonic subsidence/uplift in which the bedrock basement will moves down or up in time. With this feature, the latent heat becomes a function of both space and time.

## A brief outline of previous work

Since it publication in 2000, the Swenson analogy has been extensively applied towards understanding the formation and dynamics of sedimentary delta systems. Voller *et al.* (2004) shows that for the fixed basement geometry in Figure 3 a closed form similarity solution for tracking the movement of the shoreline can be constructed. This solution is used by Voller *et al.* (2006) to verify a numerical enthalpy-like fixed grid solution for more general cases of the problem; including the formation of two-dimensional deltas from a point sediment source. This work and more recent numerical work (Patnaik *et al.*, 2009) consider problems in which the latent heat (ocean depth) is a function of both space and time. Following up on the analytical solution of Voller *et al.* (2004), Capart *et al.* (2007) develop a range of solutions that look at alternatives to the geometry in Figure 3. The most challenging alternative geometry is the one shown in the cross-section of Figure 2(c). This problem involves tracking not one but two boundaries, the shoreline and the landward moving on-lap boundary. Independently, Lai and Capart (2009) and Lorenzo-Trueba *et al.* (2009) develop similarity solutions for this two moving boundary case; the later work arriving at a closed form solution that is validated against experimental measurements. Experimentally validated deforming grid numerical solutions for the two moving boundary problem, with the additional feature of sea-level rise, are also reported by Swenson and Muto (2007) and Parker and Muto (2003). The most recent work from Lorenzo-Trueba and Voller (2010) extends the one-dimensional two moving boundary similarity solution to cases where a nonlinear slope dependent diffusion coefficient is considered. This work also introduces a one-dimensional fixed grid enthalpy like solution for the two-moving boundary (shoreline and on-lap) problem.

The novel contribution of the current work is to develop, for the first time, a numerical solution of the two-dimensional, two-moving (shoreline and on-lap) front problem. This is the problem schematically illustrated in Figures 1 and 2. As the sediment is deposited a three-dimensional cone is formed. The plan view area of this cone (see Figure 2(b)) has two distinct moving fronts; the shoreline advancing into the ocean and the on-lap advancing landward.

## The governing equations for the example problem

Returning to the example problem defined in Figures 1 and 2. Assuming that the sediment flux on the deposited sediment surface (the fluvial surface) is described by the diffusion law:

$$\boldsymbol{q} = -\nu \nabla h, \tag{5}$$

the governing equation for the sediment mass-balance in the deposited cone is:

$$\frac{\partial h}{\partial t} = \frac{\nu}{\varepsilon} \nabla^2 h. \tag{6}$$

The domain of this equation, which changes in time, is defined by the moving shoreline and on-lap boundaries. On the shoreline boundary two conditions have to be met:

$$h = 0 \tag{7}$$

$$\nu \nabla h \cdot \boldsymbol{n} = -\varepsilon \beta x \boldsymbol{v} \cdot \boldsymbol{n}. \tag{8}$$

The first sets the shoreline sediment height at sea-level, the second is the two-dimensional form of the Swenson-Stefan condition in Equation (4) where $\boldsymbol{v}$ is the velocity of the shoreline and $\boldsymbol{n}$ is the unit normal of the shoreline pointing out of the sediment deposit.

On the on-lap boundary the conditions are:

$$h = \beta, x, \tag{9}$$

$$\nu \nabla h \cdot \boldsymbol{n} = -q \cdot \boldsymbol{n}. \tag{10}$$

The first sets the on-lap sediment height equal to the local bedrock basement height, the second accounts for the flux that is transported across the on-lap front.

A dimensionless form of the governing equations can be obtained with the introduction of the following scaling:

$$x^d = \frac{x}{\ell}, \quad y^d = \frac{y}{\ell}, \quad h^d = \frac{h}{\ell}, \quad t^d = \frac{t\nu}{\ell^2 \varepsilon}, \quad \boldsymbol{v}^d = \frac{\boldsymbol{v}\ell\varepsilon}{\nu}, \quad \boldsymbol{q}^d = \frac{\boldsymbol{q}}{\nu} \tag{11}$$

where $\ell$ is an appropriate length scale. Dropping the dimensionless superscript "*d*" for notational convenience the dimensionless form of the governing equation is:

$$\frac{\partial h}{\partial t} = \nabla^2 h \tag{12}$$

with on the shoreline boundary:

$$h = 0 \tag{13}$$

$$\nabla h \cdot \boldsymbol{n} = -\beta x \boldsymbol{v} \cdot \boldsymbol{n} \tag{14}$$

and on the on-lap boundary:

$$h = \beta x \tag{15}$$

$$\nabla h \cdot \boldsymbol{n} = -\boldsymbol{q} \cdot \boldsymbol{n}. \tag{16}$$

In this form, the problem is essentially defined by the specification of the bedrock basement slope $\beta$ and the input sediment line flux $\boldsymbol{q}_0 = (q_0, 0)$. A suitable choice for the length scale $\ell$ in Equation (11) is the width of the strip over which the sediment flux is introduced to the system, i.e. $\ell = w$. An alternative, used here, is to set $\ell$ to the side length of the square control volume cells used in the computation. In this way, the problem is fully defined by specifying the number of computational cells in the width $w$.

## The fixed grid numerical solution

In keeping with the previous numerical solutions (Voller *et al.*, 2006; Patnaik *et al.*, 2009; Lorenzo-Trueba and Voller, 2010), a fixed grid enthalpy-like solution will be developed. This solution is achieved using a cell centred control volume finite difference scheme. A grid of square control volumes $1 \times 1$ arranged as shown in Figure 4 is used. The solution is segmented into two domains:

(1) the land domain; and

(2) the ocean domain, separated by the initial position of the shoreline.

Within a time step, the solution is first constructed in the land domain and followed by advancing the solution in the ocean domain. Details in these two steps are provided.

Assuming symmetry, the land domain is defined as the region enclosed by the domain centre line, the far West boundary, the fart North boundary and the initial shoreline position. The dimensionless equation solved in the land domain is Equation (12). In solving this equation the initial condition $h = \eta$ is used, a boundary condition

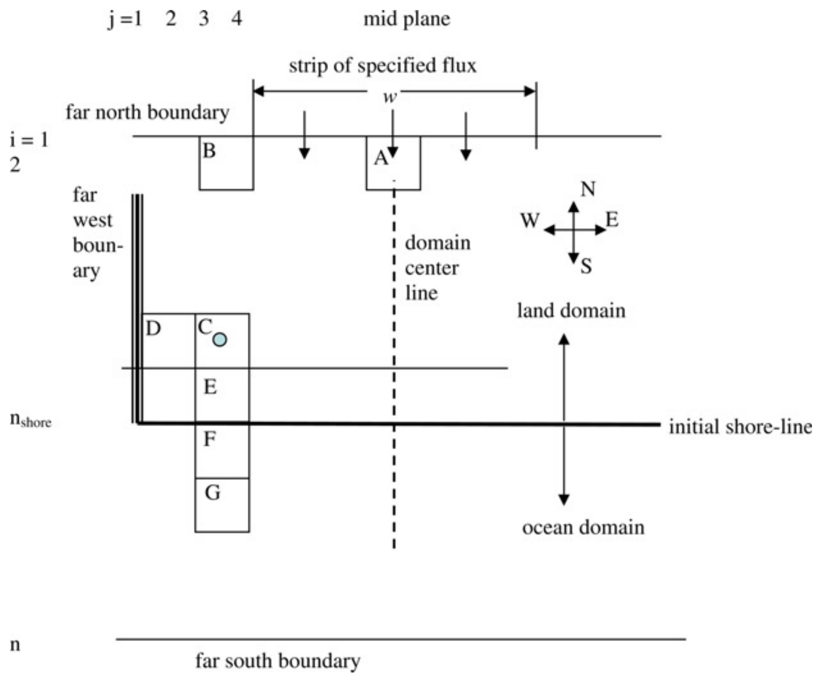



**Figure 4.**
Arrangement of domains, boundaries and control volumes

$\boldsymbol{q} = (q_0, 0)$ is applied on the far North boundary, symmetry is assumed along the domain centre line and zero sediment flux conditions are applied on the far West boundary and initial shoreline boundaries. Note:

- the physical condition $-\nabla h \cdot \boldsymbol{n} = q_0$ on the on-lap boundary (see, Equation (16)) is not explicitly applied but rather implicitly treated by determining the ability of a given control volume to balance the sediment flux entering via the diffusion transport to it neighbours; and
- the no-flow boundary on the shoreline is not physical but a computational convenience, this action is corrected in the subsequent solution over the ocean domain.

The order of the solution steps in the land domain is as follows:

(1) A solution is first constructed for Cell A (see, Figure 4) by setting the flux across the North boundary to:

$$Flux_N^{i=1} = q_0. \tag{17}$$

The flux out across the south boundary is calculated as:

$$Flux_S^i = \begin{cases} Flux_N^i, & \text{if } Flux_N^i < h(i,mid) - h(i+1,mid) \\ h(i,mid) - h(i+1,mid), & otherwise \end{cases} \tag{18}$$

$i = 1$

where *mid* is the node counter in the *y*-direction of the nodes on the domain centre line. Note with Equation (18), if the slope value of the current land surface ($h(i, mid) - h(i+1, mid)$) is too steep, the incoming prescribed flux will be by-pass through the cell without deposit; this is the gist if the implicit treatment of the on-lap condition and a similar operation will be repeated throughout the algorithm. Calculation of the fluxes associated with Cell A is completed by calculating the fluxes out across the East and West boundaries as:

$$Flux^{i}_{OUT} = [h(i, mid) - h(i, mid - 1)] + [h(i, mid) - h(i, mid + 1)] \\ i = 1. \tag{19}$$

Note this calculation will only be non-zero if sediment has been deposited on the centreline cell, i.e. if the current value $h(1, mid) > \eta(1, mid)$. With the fluxes calculated, the deposition of sediment in the cell over a time step can be explicitly calculated (assuming a dimensionless time step $\delta t = 0.1$) as:

$$h^{new}(i, mid) = h(i, mid) + 0.1\,[Flux^{i}_{N} - Flux^{i}_{S} - Flux^{i}_{OUT}] \\ i = 1. \tag{20}$$

(2) The calculations for the other cells $i > 1$ along the centreline are essentially identical to those in Equations (17)-(20). The flux across the North control volume boundary being set to the South flux of the cell immediately to the north of cell $i$, i.e. $Flux^{i}_{N} = Flux^{i-1}_{S}$ and on the initial shoreline boundary the ($i = n_{shore}$) the South flux set to zero, i.e. $Flux^{n_{shore}}_{S} = 0$.

(3) The calculations down non-centre columns $j \neq mid$ are similar to those in the centre column. The differences are: if the volume column is outside of the section where a flux is specified (see cell B in Figure 4) the initial North flux is set as: $Flux^{i=1}_{N} = 0$; the flux across the South boundary is calculated as:

$$Flux^{i}_{S} = \begin{cases} Flux^{i}_{N} + Flux^{i}_{E}, & \text{if } Flux^{i}_{N} + Flux^{i}_{E} < h(i,j) - h(i+1,j) \\ h(i, mid) - h(i+1, mid), & \text{otherwise} \end{cases} \tag{21}$$

where $Flux^{i}_{E} = h(i,j+1) - h(i,j)$ – a step that by-passes all the entering sediment through the cell if the current land surface is too steep; and the sediment update is calculated as:

$$h^{new}(i,j) = h(i,j) + 0.1\,[Flux^{i}_{N} + Flux^{i}_{E} - Flux^{i}_{S} - Flux^{i}_{W}] \tag{22}$$

where $Flux^{i}_{W} = h(i,j) - h(i,j-1)$ is the flux leaving across the West face. Note along the far West boundary (e.g. Cell D in Figure 4) the West flux is $Flux^{i}_{W} = 0$ and once again on the initial shoreline boundary (e.g. Cell E in Figure 4) the South flux is set to zero, $Flux^{n_{shore}}_{S} = 0$.

Steps 1 to 3 complete the calculations for the land nodes.

In solving for the sediment deposit in the ocean domain an enthalpy like solution is employed. In the first place an enthalpy is defined as:

$$H = h + L \tag{23}$$

where $0 < L < \beta x$ is a "latent heat" term that measures the depth of the ocean at the shoreline. At a point on the shore, while $0 < L < -\beta x$, all the sediment supplied is used to fill up the ocean and the sediment height at that point remains fixed at $h = 0$. With the definition in Equation (23), the Swenson-Stefan condition can be absorbed into a single-domain treatment with dimensionless governing equation (in the ocean domain):

$$\frac{\partial H}{\partial t} = \nabla^2 h \tag{24}$$

The initial conditions are $h = L = 0$, the boundary conditions are symmetry along the domain centre line, no flow across the far West and far South boundaries and a sediment flux specified by the previous land domain solution along the initial shoreline boundary. The key steps in the ocean domain are as follows:

(1) An initial estimate of the flux entering the North face of the ocean control volumes adjacent to the initial shoreline (e.g. Cell F in Figure 4) is calculated as:

$$Flux_N^{n_{shore}+1} = h(n_{shore}, j) - h(n_{shore} + 1, j). \tag{25}$$

This flux can only be sustained without unphysical erosion in the first landward cell at $i = n_{shore}$ if (assuming a time step of $\delta t = 0.1$):

$$Flux_N^{n_{shore}+1} \geq 10h^{new}(n_{shore}, j). \tag{26}$$

where $h^{new}(n_{shore}, j)$ is the current estimate from the land domain calculation. If the condition in Equation (26) does not hold then any sediment entering the cell at $i = n_{shore}$ by-passes directly into the ocean without land deposit, hence the North flux for the adjacent sea domain volume is reset as:

$$Flux_N^{n_{shore}+1} = 10h^{new}(n_{shore}, j) \tag{27}$$

and the previously calculated land domain sediment is readjusted to its original height, i.e. $h^{new}(n_{shore}, j) = h(n_{shore}, j) = \eta(n_{shore}, j)$. If on the other hand the condition in Equation (26) is met, the North flux for the adjacent sea domain volume retains the initial estimate from Equation (25) and the calculated land domain sediment is reset as:

$$h^{new}(n_{shore}, j) = h^{new}(n_{shore}, j) - 0.1\, Flux_N^{n_{shore}+1} \tag{28}$$

(2) With the incoming flux across the initial shoreline specified, an explicit solution for the enthalpy like equation can be advanced row by row. First, a nodal enthalpy is calculated as:

$$\begin{aligned} &H = h(i, j) + L(i, j) + 0.1\, [Flux_N^j + Flux_E^j - Flux_W^j - Flux_S^j] \\ &n_{shore} > i > n \end{aligned} \tag{29}$$

where $Flux_E^i = h(i,j+1) - h(i,j)$, $Flux_W^i = h(i,j) - h(i,j-1)$, $Flux_S^i = h(i,j) - h(i+1,j)$, and each new row is seeded by setting $Flux_N^i = Flux_S^{i-1}$.

(3) Following the calculation of the enthalpy the height of the sediment and enthalpy values can be calculated as:

$$h^{new}(i,j) = \begin{cases} H(i,j) + \eta(i,j) & \text{if } H(i,j) > -\eta(i,j) \\ 0 & otherwise \end{cases} \tag{30}$$

$$L^{new}(i,j) = \begin{cases} -\eta(i,j) & \text{if } H(i,j) > -\eta(i,j) \\ H(i,j) & otherwise \end{cases} \tag{31}$$

This step completes the time step calculations and establishes an update of the deposit sediment height field $\boldsymbol{h}^{new}$ at all nodes (land and ocean) and an updated latent heat field $\boldsymbol{L}^{new}$ in the ocean domain nodes.

Before the next time step (seeded by setting $\boldsymbol{h} = \boldsymbol{h}^{new}$ and $\boldsymbol{L} = \boldsymbol{L}^{new}$), an estimate of the plan-form (*x-y*) extent of the sediment deposit tracking the shoreline and on-lap boundaries can be made by calculating the nodal field variable:

$$c(i,j) = \begin{cases} h^{new}(i,j) - \eta(i,j) & i,j \in \text{land} - \text{domain} \\ -\mathrm{L}(\mathrm{i},\mathrm{j})/\eta(\mathrm{i},\mathrm{j}) & i,j \in \text{ocean} - \text{domain} \end{cases} \tag{32}$$

In this way, the contour $c(i,j) = 0$ marks out the plan-form area of the sediment deposit; in practice – to ensure a smoother contour – the contour $c(i,j) = 0.05$ is plotted.

## An extension for tectonic subsidence

To add a layer of complexity a simple hinge tectonic subsidence can be added to the above model. In this situation, the initial bedrock slope is $\beta_0$. As time increases, however, the basement slope in the ocean domain $x \geq 0$ increases with time according to:

$$\beta = \beta_0 + \dot{\beta} t \tag{33}$$

where $\dot{\beta}$ is a given rate of angle increase . In this model, the ocean basement is treated as a separate plate to the land bedrock basement; a plate that is allowed to hinge downward and increase in slope as the process advances in time. This form of subsidence is readily incorporated into the above model by ensuring that the basement elevation in the ocean domain is updated in time through:

$$\eta(i,j) = (\beta + \dot{\beta} t)x, \quad x > 0, \tag{34}$$

a step that makes the latent heat term in Equation (31) a function of both space and time.

## Verification

The solution above is verified in two ways. The first is simply to check the mass balance on the systems. Up to the simulation time $t$ the total sediment added to the system will be:

$$Q^{in} = q_0 wt \tag{35}$$

where $w$ is the width of the flux input strip. At the same time, the total sediment deposited is given by:

$$Q^{dep} = \sum_{i,j \in land} h(i,j) - \eta(i,j) \quad + \quad \sum_{i,j \in sea} h(i,j) + L(i,j). \tag{36}$$

If the proposed scheme conserves mass then the quantities in Equations (35) and (36) should be equal. For all the calculations conducted here, the values of $Q^{in}$ and $Q^{dep}$ are identical at every time step.

The second verification is to consider a problem where the input flux extends across the entire width of the domain. In this case, the problem becomes one dimensional, with constant x-lines representing the positions of the on-lap and shoreline fronts. If tectonic subsidence is negligible (fixed slope β), this problem is amenable to the closed form analytical solution developed by Lorenzo-Trueba *et al.* (2009). This solution sets the positions of the moving boundaries as:

$$\begin{aligned} shoreline &= 2\lambda_s\sqrt{t} \\ on-lap &= -2\lambda_o\sqrt{t}. \end{aligned} \tag{37}$$

For the values of $\beta = 0.25, \;\; q_0 = 0.2$ the values of the constants in Equations (37) are:

$$\begin{aligned} \lambda_s &= 0.666 \\ \lambda_o &= 0.44. \end{aligned}$$

The analytical solutions (lines) for the movement of the shoreline and on-lap are compared with numerical solutions (20 ($1 \times 1$) cells in the width ($y$) direction, 39 in the length ($x$) direction) in Figure 5. The numerical prediction for the shoreline front is obtained by recording the times when the latent heats along the mid-line of nodes reach the value $\eta(i, mid)/2$ ($i > n_{shore}$). The predictions for the on-lap when the mid line nodes first record sediment heights $h(i, mid) > \eta(i, mid)$ $(1 \leq i \leq n_{shore})$. The accuracy of the comparison in Figure 5 confirms the suitability of the numerical solutions steps laid out above. Note further this is also an indirect validation of the model in that Lorenzo-Trueba *et al.* (2009) have extensively validated the analytical model for tracking the front movements with experimental results.

## Results

An important application of the numerical model developed and tested above is to determine, for given scenarios, the effectiveness of delta land building in the ocean domain; land building that will mitigate the consequences of severe costal events.

First two cases of the example problem with a fixed bedrock slope $\beta = 0.25$ are examined. In both cases, grid independent results are obtained with 50 ($1 \times 1$) control volumes in the $x$-direction and 79 control volumes in the $y$-direction; the initial shoreline is placed on the South boundary of 20th row of volumes. In Case 1, the sediment flux into the systems is set as $q_0 = 0.225$ and applied over a width of seven control volumes. In

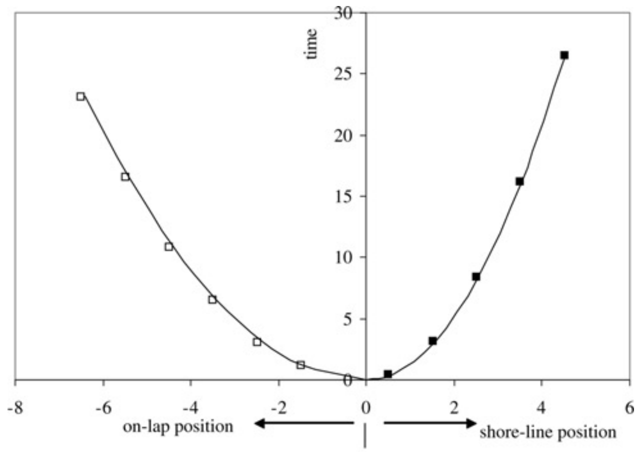
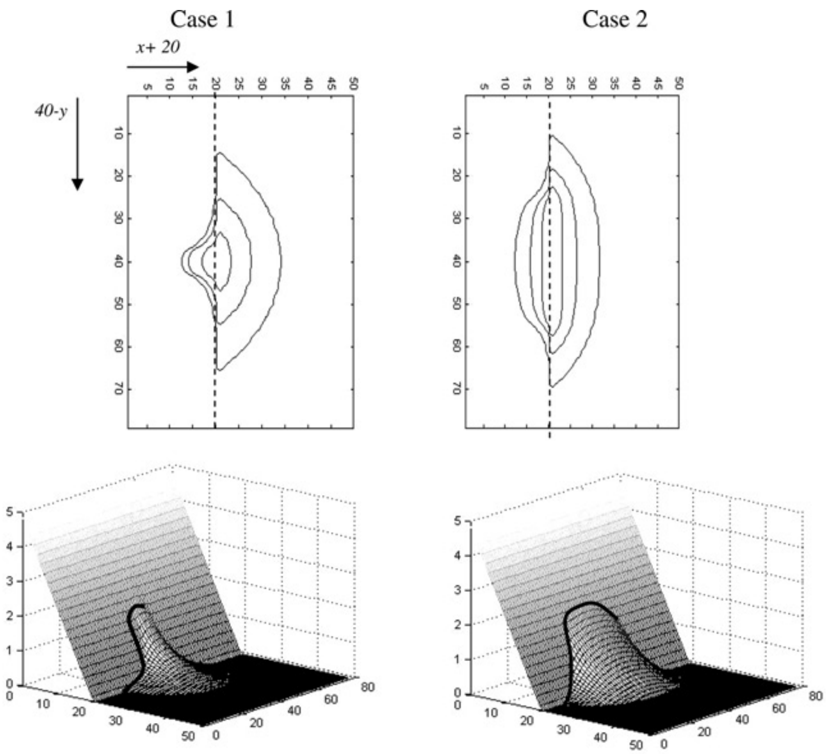



**Figure 5.**
Verification of the sediment delta model

**Notes:** Predicted movement of the on-lap and shoreline positions (symbols) with analytical solution (line)

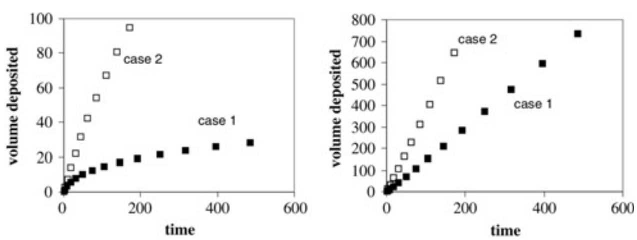
Case 2, the sediment flux is set as $q_0 = 0.15$ and applied over a width of 29 control volumes. The Case 1 result, plan-form plots of the shoreline and on-lap boundaries at times $t = 10, \quad 100, \quad 500$, and the above sea-level sediment cone at time $t = 500$ are shown in left panel of Figure 6. The Case 2 results, plan-form plots of the shoreline and on-lap boundaries at times $t = 10, \quad 50, \quad 200$, and the above sea-level sediment cone at time $t = 200$ is shown in right panel of Figure 6. A core difference between the results is that in the Case 1, with the higher flux applied over a smaller strip width, the on-lap boundary initially advances rapidly and then slows significantly, whereas the shoreline boundary has a steady advance. In contrast the plan-form shape for a small flux applied over a larger strip width (Case 2) appears to expand relatively uniformly in time in both the landward and shoreline directions. This observation is confirmed by looking at the partition of sediment mass (volume) deposited between the land and the ocean domains. The left hand side of Figure 7 shows the land volume deposit with time for both Cases 1 and 2. These curves indicate a slowing of mass (volume) deposited with time but in Case 1 (higher flux applied over a smaller width) there is a pronounced drop off in the rate of on-lap deposit as time increases. In contrast, the right hand side of Figure 7 shows the ocean deposit with time. In both cases here, the rates of mass (volume) deposited in the ocean domain are essentially constant.

As a further test case, a problem that has a hinged subsidence in the ocean domain is considered. In this problem, the plan-form domain discretization and input sediment conditions are identical to Case 1 above. The initial slope of the bedrock in both land and ocean domains is $\beta = 0.25$. As time increase, however, the bedrock in the sea domain hinges downward so that at time $t$ its slope is $0.25(1 + t/500)$. In this way, over the simulation time $0 \leq t \leq 500$, the ocean bedrock slope doubles in value. In Figure 8, the predicted plan-form positions of the on-lap and shoreline fronts at times $t = 10, \ 100, \ 500$ with this subsidence are compared with the no-subsidence predictions of Case 1. The effect of the increasing ocean depth is quite clear. The on-lap front, evolving in an area without subsidence, is not significantly affected by the hinge subsidence in the ocean domain. At early




**Notes:** Case 1 high sediment flux applied over a small width, plan-form locations of the on-lap and shoreline boundaries at times ($t = 10, 100, 500$) and above sea-level deposit at time $t = 500$. Case 2 low sediment flux applied over a large width, plan-form locations of the on-lap and shoreline boundaries at times ($t = 10, 50, 200$) and above sea-level deposit at time $t = 200$

**Figure 6.**
Two simulation cases for the deposit of a sediment delta onto a constant sloping bedrock




**Notes:** The left hand panel shows on-lap deposition rates the right hand panel shows shoreline deposition rates

**Figure 7.**
Rate of sediment volume deposits in simulation Cases 1 and 2

simulation times ($t = 10, 100$), when the additional accommodation created by the hinge subsidence is still relatively small, the positioning of the shoreline is close to that seen in the non-subsiding case. At late simulation times ($t = 500$), however, the increased accommodation space results in a significantly smaller plan-form area for the delta in the ocean domain.

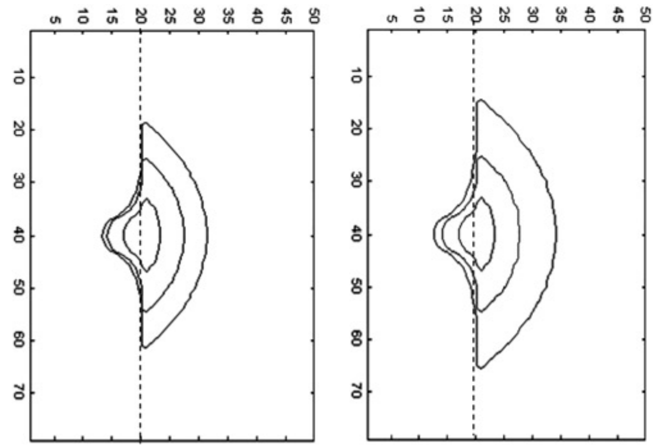


**Figure 8.**
Comparison of plan form delta deposition when a hinged subsidence is presents (left hand frame) and when it is not (right hand frame)

## Conclusions

The application of numerical heat transfer methods originally developed for free and moving boundary problems has proved to be of great benefit in advancing the understanding of the formation of sedimentary deltas. In a general setting, the mass balance problem in delta formation can be viewed as a generalized the Stefan problem with the ocean bathymetry taking the role of the latent heat. The distinctive feature in this class of problems, that test basic heat transfer the Stefan problem methods, is that the latent heat can be a function of both space (a variable ocean bathymetry) and time (tectonic subsidence/uplift or sea-level fall/rise). In the past, the majority of heat transfer applications for modelling delta growth have focused on the deposit of a two-dimensional sediment wedge in a one-dimensional domain bounded below by a specified bedrock basement. In this work, the deposit of a three-dimensional sediment cone onto a sloping bedrock plain entering an ocean is considered. The unique numerical challenge in this problem is to develop methods that can track two moving fronts, the seaward advancing shoreline and the on-lap boundary marking the landward advance of the sediment deposit. A fixed grid enthalpy like solution for this problem has been successfully developed and verified. Predictions from the resulting numerical model throw light on the critical question of how much of the sediment entering the system is used to advance the shoreline and thereby provide protection of costal communities from natural calamities.

In closing it is noted that, although understanding the role of the sediment mass balance is a critical first step towards the prediction of delta growth to complete the picture other important elements have to be included in the model. In particular, it needs to be recognized that the sediment is transported over the delta surface via water flows. In the current mass balance model, it is implicitly assumed that the water flow either covers the delta at all times (sheet flow) or that the flow channel switching over the delta surface is at faster rate that the sediment deposition process. In more refined models, it will be necessary to explicitly account for the formation and dynamics of the channels on the delta surface. Not only will this add to the resolution of the mass transport and deposition it will also provide key information for determining the plant succession essential for stabilizing the land building. Further, once plants are included, it also becomes important to consider the role of peat production and decay in the land building process.

Delta building models that resolve channels and account for net biotic production are current areas of research with the National Center for Earth Surface Dynamics. As these models develop researchers in this centre continue to look towards the application of heat transfer techniques to deal with these more refined modelling issues. For example, an exploration between dendritic crystal growth models (Voller, 2008) and delta channel formation.

**References**

Capart, H., Bellal, M. and Young, D.L. (2007), "Self-similar evolution of semi-infinite alluvial channels with moving boundaries", *Journal of Sediment Research*, Vol. 77, pp. 13-22.

Crank, J. (1984), *Free and Moving Boundary Problems*, Clarendon Press, Oxford.

Day, J.W. Jr, Boesch, D.F., Clairain, E.J., Kemp, G.P., Laska, S.B., Mitsch, W.J., Orth, K., Mashriqui, H. Reed, D.J., Shabman, L., Simenstad, C.A., Streever, B.J., Twilley, R. R., Watson, C.C., Wells, J.T. and Whigham, D.F. (2007), "Restoration of the Mississippi delta: lessons from hurricanes Katrina and Rita", *Science*, Vol. 315, pp. 1679-84.

Kim, W., Mohrig, D., Twilley, R., Paola, C. and Parker, G. (2009), "Is it feasible to build new land in the Mississippi river delta?", *EOS, Transactions American Geophysical Union*, Vol. 90 No. 42, p. 373.

Lai, S.Y.J. and Capart, H. (2009), "Reservoir infill by hyperpycnal deltas over bedrock", *Geophysical Research Letters*, Vol. 36, p. L08402.

Lorenzo-Trueba, J. and Voller, V.R. (2010), "Analytical and numerical solution of a generalized Stefan problem exhibiting two moving boundaries with application to ocean delta formation", *Journal of Mathematical Analysis and Applications*, Vol. 366, pp. 538-49.

Lorenzo-Trueba, J., Voller, V.R., Muto, T., Kim, W., Paola, C. and Swenson, J.B. (2009), "A similarity solution for a dual moving boundary problem associated with a coastal-plain depositional system", *The Journal of Fluid Mechanics*, Vol. 628, pp. 427-43.

Paola, C. and Voller, V.R. (2005), "A generalized Exner equation for sediment mass balance", *Journal of Geophysical Research*, Vol. 110, p. F04014.

Paola, H.C., Heller, P.L. and Angevine, C.L. (1992), "The large-scale dynamics of grain-size variation in alluvial basins. Part 1. Theory", *Basin Research*, Vol. 4, pp. 73-90.

Parker, G. and Muto, T. (2003), "One-dimensional numerical model of delta response to rising sea-level", *Proceedings of the Third IAHR Symposium, River, Coastal and Estuarine Morphodynamics*, *Barcelona*, pp. 558-70.

Patnaik, S., Voller, V.R., Parker, G. and Frascati, A. (2009), "Morphology of a melt front under a condition of spatial varying latent heat", *International Communications Heat and Mass Transfer*, Vol. 36, pp. 535-8.

Swenson, J.B. and Muto, T. (2007), "Response of coastal plain rivers to falling relative sea-level: allogenic controls on the aggradational phase", *Sedimentology*, Vol. 54, pp. 207-21.

Swenson, J.B., Voller, V.R., Paola, C., Parker, G. and Marr, J.G. (2000), "Fluvio-deltaic sedimentation: a generalized Stefan problem", *European Journal of Applied Mathematics*, Vol. 11, pp. 433-52.

Voller, V.R. (2008), "An enthalpy method for modeling dendritic growth in a binary alloy", *International Journal of Heat and Mass Transfer*, Vol. 51, pp. 823-34.

Voller, V.R., Swenson, J.B. and Paola, C. (2004), "An analytical solution for a Stefan problem with variable latent heat", *International Journal of Heat and Mass Transfer*, Vol. 47, pp. 5387-90.

Voller, V.R., Swenson, J.B., Kim, W. and Paola, C. (2006), "An enthalpy method for moving boundary problems on the earth's surface", *International Journal of Numerical Methods for Heat and Fluid Flow*, Vol. 16, pp. 641-54.

**Further reading**

Kim, W. and Muto, T. (2007), "Autogenic response of alluvial-bedrock transition to base-level variation: experiment and theory", *Journal of Geophysical Research*, Vol. 112, p. F03S14.

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